*International Journal of Theoretical Physics,* Vol. 15, No. 12 (1976), pp. 927–947

# Bases for a Discrete Special Relativity[1]

M. LORENTE[2]

*Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts*

*Received: 27 October 1975*

*Abstract*

Following recent developments in the hypothesis of a discrete space-time lattice, some assumptions are postulated that seem necessary to work out this model in the theory of special relativity. In particular, the assumption of space-time coordinates with integer values requires the translation of relativistic mechanics and electrodynamics into the language of finite difference equations. A special study of the covariance of these equations under the inhomogeneous Lorentz group is carried out. Finally, a stronger assumption is postulated, by which the physical magnitudes derived from the space-time coordinates should take rational values.

## 1. *Introduction*

From time to time the idea of a discrete space-time has attracted the attention of physicists. Recently Greenspan (1973) in several papers and a book has worked out methods of numerical calculations suitable for computer programs, based on the idea of a discrete mechanics. At the end of his book he challenges other scientists to carry out an extensive program about discrete models, among them a complete study of a discrete special relativity.

The idea of physical measurements led Taylor and Wheeler (1966) to assume a discrete cubic lattice where all the physical events take place, with space-time coordinates given by integral numbers.

In order to describe the quark confinement for strong coupling, K. G. Wilson (1974a)[3] introduced a discrete space-time lattice, which gives a finite energy proportional to the separation of quarks. Nevertheless, he recognized that his theory is far from covariant owing to the lattice.

1 This work is supported in part through funds provided by the Atomic Energy Commission under Contract No. AT11-1-3069.

2 Sponsored by G.I.F.T. Permanent address: Departamento de Física Teórica, Facultad de Ciencias, Universidad de Madrid, Madrid-3.

3 I am thankful to Professor R. Jackiw for bringing to my attention Wilson's ideas.

In a recent paper, we have also tried to describe a physical world, the structure of which is supported by a (3 + 1)-dimensional cubic lattice (Lorente, 1974). More references can be found in this paper about other people who have approached this classic problem.

The way of starting this theory, which seems to be arduous from the beginning, has not been unanimous at all. The weakest assumption states discreteness in the eigenvalues of the space and time operators, following the principles of quantum mechanics. What looks more radical is a stronger assumption that proposes discrete space-time variables though the functions representing physical quantities can be represented by continuous functions. The strongest assumption tries to extend the discreteness to all the physical magnitudes, not only space-time, even at the risk of running into a dead end.

In this paper we work out the possibility of a discrete special relativity, using the assumption of a discrete space-time, which requires that the most fundamental formulas of relativistic mechanics and electrodynamics be translated into the discrete language. Special care will be taken with respect to the problem of covariance. It turns out that the equations of mechanics are easily written in a covariant form. The situation in classical electrodynamics looks more difficult in principle. At least in one particular case, covariance is obtained as a result of auxiliary conditions.

To many people the idea of a discrete space-time seems unfounded. It can be true, but the bases for a continuous space-time also lacks any serious foundations. It is not unreasonable to try, as a hypothesis, all the possible approaches. They may be contradicted by experimental results, or they may lead to some superselection rules useful to describe unexplained phenomena.

## 2. *General Assumptions*

In order to describe the fundamental laws of physics in a discrete form it is important to know from the beginning which particular postulates of the special theory of relativity are retained and which of them are modified. In general, we can say that only those principles that can be applied to discrete properties of matter should be kept. Therefore, the two fundamental assumptions of special relativity remain the same:

(1) All physical phenomena will take the same form in any system of inertia, in other words, all systems of inertia are equivalent.

(2) The propagation of velocity of light in empty space must have the same constant value in every system of inertia.

The second assumption follows from the first one when we apply it to the Maxwell equations; however, since we are going to modify these equations we prefer to make a different assumption.

The new assumption we introduce is connected with the discrete properties of matter. As far as our observations are concerned we can only measure finite values of physical quantities such as lengths, time intervals, masses, charges, and so on. Therefore we do not lose any kind of generality if we describe the physical laws with discrete magnitudes. Since the fundamental laws of physics have been proved to be very successful in their differential form it is straightforward to combine the aforementioned properties in the following assumption:

(3) The expression for the physical laws in special relativity should retain the same relations if the values of the magnitudes involved are finite, but the local properties of these magnitudes, when they appear, should be described by discrete quantities.

This assumption requires the use of numerical calculus, which is better suited to computer program than the functional analysis (Greenspan, 1973). Moreover, if the finite increments of the magnitudes can only take integer values, then the description of physical laws should be made in terms of difference equations, instead of the classical form of the differential equations.

## 3. *Relativistic Kinematics*

3.1. *Coordinates of an Event.* A very good way to visualize the finite measurements of some physical quantity is the cubical latticework of meter sticks, described by Taylor and Wheeler (1966), representing one inertial frame. At every intersection of the lattice there is a clock, and all the clocks are synchronized with one another. Although in the Taylor–Wheeler picture two neighboring intersections are separated by one meter in distance and one meter of light in time, one can choose smaller units to have a more precise picture.

In order to determine the location and instant of some event, we can take the position and time of the clock nearest to the event. If we have chosen in advance some origin of spacial coordinates to distinguish each individual clock and some instant as the initial time for all the synchronized clocks, we can take as coordinates of the event the spacial position and time of the nearest clock with respect to the chosen origin, The coordinates of the event will be represented by the usual four-vector

$$x \equiv (x_1, x_2, x_3, x_4), \qquad x_4 = ct \tag{3.1}$$

where $c$ is the velocity of light in empty space.

The same event can be described by another inertial frame, which is visualized by a different latticework of meter sticks and synchronized clocks. The coordinates of the event with respect to this new reference system will be

$$x' \equiv (x_1', x_2', x_3', x_4'), \qquad x_4' = ct' \tag{3.2}$$

It can be argued that this way of location for one event is not precise, even if the distance between clocks becomes very small. Although this is true from the point of view of a continuous model of space and time, the quantum theory of measurement forbids us to get a better accuracy beyond certain limits. Nevertheless this naive description of inertial systems will be considered completely exact from the conception of a discrete space-time.

3.2. *Transformation of Coordinates.* The next step in the construction of a discrete relativistic kinematics is to find out the transformations of coordinates of some event, which are consistent with the assumptions stated before. It is very well known that the most general transformations that satisfy assumptions (1) and (2) are given by the elements of the proper Lorentz group. (In classical physics we are not concerned with parity and time reversal transformations.) But it can be checked that in the derivation of Lorentz formula no use has been made of infinitesimal values of the coordinates or local properties of the group. Therefore, they are consistent with assumption (2).

The fundamental representation of the Lorentz group can be made with the help of Euler angles as the parameters of the group. We want to reproduce here another representation using Cayley parameters as explained in Lorente (1974). For special Lorentz transformations of the coordinates in the system $S'$ with respect to the coordinates of the same event in other system $S$ we have

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} m^2+r^2-s^2-t^2 & 2rs & 2rt & -2mr \\ 2rs & m^2-r^2+s^2-t^2 & 2st & -2ms \\ 2rt & 2st & m^2-r^2-s^2+t^2 & -2mt \\ -2mr & -2ms & -2mt & m^2+r^2+s^2+t^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{3.3}$$

where $\Delta = m^2 - r^2 - s^2 - t^2$ must be positive in order that the transformation belong to the proper Lorentz group. The relations between the Cayley parameters $(m, r, s, t)$ and the relative velocity $\mathbf{v}$ of the two inertial systems $S$ and $S'$ are given by

$$v_1 = \frac{2cmr}{m^2+r^2+s^2+t^2}, \qquad v_2 = \frac{2cms}{m^2+r^2+s^2+t^2}, \qquad v_3 = \frac{2cmt}{m^2+r^2+s^2+t^2} \tag{3.4}$$

From this it follows that $v^2 \leqslant c^2$; the equality holding only when $r^2+s^2+t^2 = m^2$.

For a rotation of the system $S$ with respect to the system $S'$ the coordinates of the same event will transform as follows:

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} m^2-n^2-p^2+q^2 & -2mn+2pq & 2mp+2nq \\ 2mn+2pq & m^2-n^2+p^2-q^2 & -2mq+2np \\ -2mp+2nq & 2mq+2np & m^2+n^2-p^2-q^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{3.5}$$

where $\Delta = m^2 + n^2 + p^2 + q^2$. The relation between the Cayley parameters $(m, n, p, q)$ and the axis $\mathbf{a}$ and angle of rotation $\sin\phi = a = |\mathbf{a}|$ is given by

$$a_1 = \frac{2mq}{m^2+n^2+p^2+q^2}, \qquad a_2 = \frac{2mp}{m^2+n^2+p^2+q^2}, \qquad a_3 = \frac{2mn}{m^2+n^2+p^2+q^2} \tag{3.6}$$

The geometrical interpretation of the Cayley parameters in the pure Lorentz transformations and three-dimensional rotations coincides with the interpretation of the classical Euler parameters only when the last ones correspond to infinitesimal transformations (see, for instance, Fock, 1964). For this reason the Cayley parameters are suitable to describe finite transformations as well as very small ones, provided that, in the latter case, the parameters $n, p, q$ or $r, s, t$ are very small in comparison with $m$.

Finally for a general transformation of the proper Lorentz group we have

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} m^2 - n^2 - p^2 + q^2 + r^2 - s^2 - t^2 - \lambda^2 & -2mn + 2pq + 2rs + 2\lambda t & 2mp + 2nq + 2rt - 2\lambda s & -2mr + 2ns - 2pt - 2\lambda q \\ 2mn + 2pq + 2rs - 2\lambda t & m^2 - n^2 + p^2 - q^2 - r^2 + s^2 - t^2 + \lambda^2 & -2mq + 2np + 2st + 2\lambda r & -2ms - 2nr + 2qt - 2\lambda p \\ -2mp + 2nq + 2rt + 2\lambda s & 2mq + 2np + 2st - 2\lambda r & m^2 + n^2 - p^2 - r^2 - s^2 + t^2 + \lambda^2 & -2mt + 2pr - 2qs - 2\lambda n \\ -2mr - 2ns + 2pt - 2\lambda q & -2ms + 2nr - 2qt - 2\lambda p & -2mt - 2pr + 2qs - 2\lambda n & m^2 + n^2 + p^2 + q^2 + r^2 + s^2 + t^2 + \lambda^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{3.7}$$

where

$$m\lambda = nt + ps + qr$$

and

$$\Delta = m^2 + n^2 + p^2 + q^2 - r^2 - s^2 - t^2 - \lambda^2$$

The general transformation (3.7) will be summarized from now on by the standard notation[4],

$$x_\mu' = \Lambda_\mu{}^\nu x_\nu \qquad (\mu, \nu = 1, 2, 3, 4) \tag{3.8}$$

We have to consider also the transformation of coordinates with respect to translations, that is to say,

$$x_1' = x_1 + a_1$$
$$x_2' = x_2 + a_2$$
$$x_3' = x_3 + a_3$$
$$x_4' = x_4 + a_4$$

which in standard notation reads

$$x_\mu' = x_\mu + a_\mu \tag{3.9}$$

In order that the discreteness of space-time be conserved the general element of the proper Lorentz group (3.7) must have integer components, which is met if

$$\Delta \equiv m^2 + n^2 + p^2 + q^2 - r^2 - s^2 - t^2 - \lambda^2 = 1 \tag{3.10}$$

Obviously these elements form a subgroup. [The only nontrivial property to be proved is that every element has an inverse that belongs to the same subgroup, but the inverse element is represented by the parameters ($m, -n, -p, -q, -r, -s, -t$), which satisfies (3.10), when the original element does.]

[4] I use the metric $g_{11} = g_{22} = g_{33} = -g_{44} = 1$.

For the translation group, the discreteness condition is easily met, if the parameters take only integral values:

$$a_\mu = \text{integer} \tag{3.10a}$$

If we combine the Lorentz group and the translations, as a semidirect group, the elements that satisfy the discreteness condition form a subgroup. As before, the only nontrivial property to be proved is the existence of the inverse element. But given a general element of the Poincaré group $(a_\mu, \Lambda_\mu{}^\nu)$, the parameters of which satisfy (3.10) and (3.10a), the corresponding inverse element $(-\Lambda_\nu{}^\mu a_\mu, \Lambda_\nu{}^\mu)$ belongs also to the same subgroup.

3.3. *The Interval of Two Events.* Consider two different events $\{1\}$ and $\{2\}$, whose coordinates with respect to two inertial frames $S$ and $S'$ are labeled by

$$\{x_\mu{}^1\},\ \{x_\mu{}^2\} \text{ and } \{x_\mu'^1\},\ \{x_\mu'^2\} \tag{3.11}$$

Suppose that both systems $S$ and $S'$ are related by some inhomogeneous Lorentz transformations. According to (3.8) and (3.9) we must have

$$x_\mu'^{(\kappa)} = \Lambda_\mu{}^\nu x_\nu^{(\kappa)} \ (\kappa = 1, 2) \tag{3.12}$$

$$x_\mu'^{(\kappa)} = x_\mu^{(\kappa)} + a_\mu \ \ (\kappa = 1, 2) \tag{3.13}$$

We construct the finite differences between the components of the two events,

$$\Delta x_\mu \equiv x_\mu{}^1 - x_\mu{}^2 \tag{3.14}$$

Under a homogeneous Lorentz transformation this four-vector becomes from (3.12)

$$\Delta x_\mu' = \Lambda_\mu{}^\nu \Delta x_\nu \tag{3.15}$$

and under a translation given by (3.13)

$$\Delta x_\mu' = \Delta x_\mu \tag{3.16}$$

From (3.15) and (3.16) it follows that the norm of the four-vector $\{\Delta x_\mu\}$ is invariant under an inhomogeneous Lorentz transformations (3.8) and (3.9)

$$(\Delta x_1')^2 + (\Delta x_2')^2 + (\Delta x_3')^2 - (\Delta x_4')^2 = (\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2 - (\Delta x_4)^2 \tag{3.17}$$

This invariant is called the *interval* between the two events. According to the standard classification, the vector $\{\Delta x_\mu\}$ will be spacelike, timelike, or light-like when the interval is positive, negative, or zero. In particular the timelike interval

$$\Delta\tau = (1/c)[(\Delta x_4)^2 - (\Delta x_1)^2 - (\Delta x_2)^2 - (\Delta x_3)^2]^{1/2} \tag{3.18}$$

is called the *proper time* between the two events.

3.4. *Transformation of Velocities.* According to the assumption (3), the velocity of a particle should be described by discrete quantities. Given a particle that is moving with respect to some reference system, the position and time in two consecutive events of the same particle are represented by the four-vectors $\{x_\mu{}^1\}$ and $\{x_\mu{}^2\}$. It is natural to define the three-vector velocity as the quotient of the spacial and time displacement (Lorente, 1974, p. 241):

$$u_1 = \frac{\Delta x_1}{\Delta t}, \qquad u_2 = \frac{\Delta x_2}{\Delta t}, \qquad u_3 = \frac{\Delta x_3}{\Delta t} \tag{3.19}$$

where $\Delta x_1$, $\Delta x_2$, $\Delta x_3$, $\Delta t$ are discrete quantities. Other authors prefer to define the three-velocity in different ways for different reasons.[5] We prefer (3.19) because, as we will prove, it is more consistent with covariant considerations.

The transformations of the three components of the velocity under a Lorentz transformation are obtained immediately from (3.15):

$$u'_k = \frac{\Delta x'_k}{\Delta t'} = c\frac{\Lambda_k{}^\nu \Delta x_\nu}{\Lambda_4{}^\nu \Delta x_\nu} = \frac{\Lambda_k{}^m c u_m + \Lambda_k{}^4 c^2}{\Lambda_4{}^m u_m + \Lambda_4{}^4 c} \qquad (k, m = 1, 2, 3) \tag{3.20}$$

which can be easily put in vectorial form using the three components $(v_1, v_2, v_3)$ of the relative velocity in the case of a pure Lorentz transformation (see Møller, 1952).

Obviously the transformations of the three-velocity (3.19) under translations (3.9) are given from (3.16):

$$u'_1 = u_1, \qquad u'_2 = u_2, \qquad u'_3 = u_3 \tag{3.21}$$

Let us consider the motion of a particle in the (3 + 1)-dimensional Minkowski space. The line described by this motion will be called the world-line of the particle. We can use the length of this line as the parameter for the representation of this motion, since it is an invariant under the inhomogeneous Lorentz group

$$\Delta s = (\Delta x_\mu \Delta x^\mu)^{1/2} = c\Delta\tau \tag{3.22}$$

where $\tau$ is the proper time of the particle.

If we define the four-velocity as

$$U_\mu = \Delta x_\mu / \Delta\tau \tag{3.23}$$

from (3.22) we obtain

$$U_\mu U^\mu = c^2 \tag{3.24}$$

and taking the finite difference of the last equation, we get

$$(\Delta U_\mu) U^\mu + U_\mu \Delta U^\mu + \Delta U_\mu \Delta U^\mu = 0 \tag{3.25}$$

[5] Greenspan (1973) has discussed different definitions of the three-velocity. Although he uses our definition (3.19) in the demonstration of the relativistic mass formula, he prefers a different one for the nonrelativistic case

The last term $\Delta U_\mu \Delta U^\mu$ is an invariant. In order to know its value, we take an inertial frame where the three-vector $\Delta U_\kappa = 0$. Since

$$U_k = \frac{u_k}{(1 - u^2/c^2)^{1/2}}$$

in this particular inertial frame the velocity $\mathbf{u}$ remains constant in the two consecutive events. Therefore

$$U_4 = \frac{c}{(1 - u^2/c^2)^{1/2}}$$

also does not change in the two consecutive events. Therefore $\Delta U_4 = 0$ and $\Delta U_\mu \Delta U^\mu = 0$. Then dividing (3.25) by $\Delta\tau$ we obtain

$$U_\mu \Delta U^\mu / \Delta\tau = 0 \tag{3.26}$$

a result which was proved in the classical definition of velocity. The four-vector

$$\Delta_\mu = \Delta U^\mu / \Delta\tau \tag{3.27}$$

is called the four-acceleration.

Another expression which can be written without modification, is the wave number four-vector of a plane monochromatic wave:

$$K_\mu = 2\pi \left( \frac{\bar{n}}{\lambda}, \frac{1}{T} \right) \tag{3.28}$$

where $\lambda$ is the wavelength of the wave, $T$ its period, and $\bar{n}$ a unit vector in the direction of the wave normal. It is well known that the phase of the wave is an invariant under Lorentz transformations

$$K'_\mu x'^\mu = K_\mu x^\mu \tag{3.29}$$

and therefore the wave number four-vector transforms as

$$K'_\mu = \Lambda_\mu{}^\nu K_\nu \tag{3.30}$$

There is an obvious consequence of the kinematical properties of this model. The motion of a particle, if it is not continuous, should be described by discrete jumps on the points of a (3 + 1)-dimensional cubic lattice. Between consecutive events the proper time given by (3.18) will not take, in general, the same values. Therefore it is not a convenient parameter as in the continuous case. The most useful way to describe a discrete motion seems to be with the help of a parameter labeling the sequence of events. Thus the covariant expression for the motion of a particle will be

$$x_\mu = x_\mu(u)$$

where $x_\mu(u)$ are entire functions of the variable $u$, which can take only integer values, and is invariant under inhomogeneous Lorentz transformations. This kinematical variable was suggested from different considerations by Aghassi et al. (1970) as mentioned in Lorente (1974).

## 4. *Relativistic Mechanics*

4.1. *Momentum and Energy of a Particle.* In order to derive the expression of the relativistic three-momentum of a particle of mass $m_0$ the classical arguments are based on the assumptions of special relativity [assumptions (1) and (2)] and the more general one of the conservation of momentum. If we impose in this derivation an extra condition of discreteness [assumption (3)] we can retain the same expression for the momentum, provided that in its calculation no infinitesimal properties for the motion of the particle are used. It is easily checked that starting with the definition of the three-velocity $\mathbf{u} = \Delta\mathbf{x}/\Delta t$, the same classical arguments together with the assumption of discreteness will lead to the definition of three-momentum

$$\mathbf{p} = \frac{m_0 \mathbf{u}}{(1 - u^2/c^2)^{1/2}} = m_0 \frac{\Delta \mathbf{x}}{\Delta \tau} \tag{4.1}$$

Now we want a fourth component of the momentum in order to have a four-momentum, in such a way that it transforms covariantly under the inhomogeneous Lorentz group.

Following an elegant argument by Taylor and Wheeler (1966, pp. 111–114), we rewrite the transformations (3.15), multiply both sides by $m_0$, and divide by $\Delta\tau$ (recall that $\Delta\tau = \Delta\tau'$):

$$m_0 \frac{\Delta x'_\mu}{\Delta \tau'} = \Lambda_\mu{}^\nu m_0 \frac{\Delta x_\nu}{\Delta \tau} \tag{4.2}$$

The objects so constructed, $m_0 \Delta x_\mu/\Delta\tau$, transform as the components of a four-vector. But the first three components of it are exactly the components of the three-momentum (4.1). Therefore we can define the four-momentum as

$$p_\mu = m_0 \, \Delta x_\nu/\Delta\tau \tag{4.3}$$

The fourth component of this vector is

$$p_4 = m_0 \frac{\Delta x_4}{\Delta \tau} = \frac{m_0 c}{(1 - u^2/c^2)^{1/2}} \tag{4.4}$$

which, multiplied by $c$, is the relativistic energy of the particle, in discrete form. There are several reasons to call $p_4 c = E$ the relativistic energy. First of all, in the nonrelativistic limit, $u \ll c$,

$$E \sim m_0 c^2 + \tfrac{1}{2} m_0 v^2 \tag{4.5}$$

$E$ goes to the sum of the rest energy and its kinetic energy.[6]

Secondly $E$ satisfies the same conservation laws as the classical energy. The argument is based only on the conservation of three-momentum and is consistent with the assumption of discreteness.[7] In fact, from (4.2) we have

$$p'_\mu = \Lambda_\mu{}^\nu p_\nu \tag{4.6}$$

Suppose we have an elastic collision of two particles, with total four-momentum $p_\mu$ and $q_\mu$, before and after the collision, respectively. Under a Lorentz transformation, $q_\mu$ behaves as a four-vector:

$$q'_\mu = \Lambda_\mu{}^\nu q_\nu \tag{4.7}$$

Obviously, the difference of both four-momenta $\Delta p_\mu = p_\mu - q_\mu$ is transformed as

$$\Delta p'_\mu = \Lambda_\mu{}^\nu \Delta p_\nu \tag{4.8}$$

The first three equations of (4.8) read

$$\Delta p'_k = \Lambda_k{}^j \Delta p_j + \Lambda_k{}^4 \Delta p_4 \qquad (j, k = 1, 2, 3) \tag{4.9}$$

from the assumption of conservation of momentum, valid in every inertial system, $p_k = q_k$, and $p'_k = q'_k$, hence $\Delta p_k = \Delta p'_k = 0$, which from (4.9) gives $\Delta p_4 = 0$. Taking this result in the fourth equation of (4.8) we get

$$\Delta p'_4 = \Lambda_4{}^k \Delta p_k + \Lambda_4{}^4 \Delta p_4 = 0 \tag{4.10}$$

Therefore $p_4 = q_4$ and $p'_4 = q'_4$, which means that the fourth component of the momentum is conserved in every inertial system.

4.2. *The Norm of the Four-Momentum.* From (4.1) and (4.4) we easily obtain

$$\frac{\mathbf{p}}{E} = \frac{1}{c^2}\frac{\Delta \mathbf{x}}{\Delta t} \tag{4.11}$$

We want now a relation between the finite increments of the three-momentum and energy of a particle. We take the finite difference of the Lorentz invariant

$$\mathbf{p}^2 - E^2/c^2 = -m_0{}^2 c^2 \tag{4.12}$$

which follows from (4.3), and remembering the rule for the finite difference of the product of two functions, we have

$$2\mathbf{p}\Delta\mathbf{p} + (\Delta\mathbf{p})^2 - (1/c^2)[2E\Delta E + (\Delta E)^2] = 0 \tag{4.13}$$

The difference of momentum in two consecutive events of the particle, $\Delta p_\mu$, is also a four-vector, therefore its norm $\Delta p_\mu \Delta p^\mu$ is a Lorentz invariant. In order to calculate this invariant we take an inertial system such that $\Delta\mathbf{p} = 0$, which means that the three-momentum of the particle is the same for the two consecutive events, and consequently the velocity and energy are also the same. Therefore, in this particular frame $\Delta\mathbf{p} = \Delta E = 0$, and so in an arbitrary inertial frame

$$(\Delta\mathbf{p})^2 - (1/c^2)(\Delta E)^2 = 0$$

[6] Greenspan has proved (1973, p. 136) that using the discrete expression for the relativistic energy one can prove the relation $E = mc^2$.

[7] This argument follows Taylor and Wheeler's book (1966).

Inserting this result in (4.13) and using (4.11), we obtain

$$\Delta E = \mathbf{u} \cdot \Delta \mathbf{p} \tag{4.14}$$

4.3. *Force and Equation of Motion.* The definition of force is made up in special relativity with the help of the relativistic three-momentum. We should keep an equivalent definition in case we want to introduce assumption (3), but then we have to take the definition of four-momentum in the discrete sense, given by (4.1) and (4.4). Therefore, the three-component force is defined as

$$\mathbf{F} = \Delta \mathbf{p}/\Delta t \tag{4.15}$$

This equation is not only the definition of the force but the equation of motion in a discrete relativistic mechanics.

Multiplying both sides of (4.15) by $\mathbf{u}$ and using (4.14), we obtain

$$\mathbf{F} \cdot \mathbf{u} = \Delta E/\Delta t \tag{4.16}$$

which expresses in discrete form the equivalence between the work done by the force and the change of the total energy per unit time. This is also another reason to call energy the fourth component of the formally constructed four-vector $p_\mu$, as defined in (4.3).

Equations (4.15) and (4.16) are still not in covariant form. In order to do it, we divide both sides by $(1 - u^2/c^2)^{1/2}$. The four equations

$$\frac{\Delta \mathbf{p}}{\Delta \tau} = \frac{E}{(1 - u^2/c^2)^{1/2}}, \qquad \frac{\Delta E}{\Delta \tau} = \frac{\mathbf{F} \cdot \mathbf{u}}{(1 - u^2/c^2)^{1/2}} \tag{4.17}$$

are now covariant, since the numerators in the left side transform as a four-vector and the denominator is a Lorentz invariant. Therefore, the right side must be also the components of a four-vector. This four-force, or Minkowski force, is defined as

$$F_\mu = \left( \mathbf{F} \frac{\Delta t}{\Delta \tau}, \frac{\mathbf{F} \cdot \Delta \mathbf{x}}{c \Delta \tau} \right) \tag{4.18}$$

and (4.17) can be rewritten in the form

$$\Delta p_\mu / \Delta \tau = F_\mu \tag{4.19}$$

or in terms of the four-velocity (3.23)

$$m_0 \frac{\Delta U_\mu}{\Delta \tau} = F_\mu \tag{4.20}$$

Contracting both sides with $U^u$, and using (3.26), we obtain

$$F_\mu U^\mu = 0 \tag{4.21}$$

a result equivalent to the continuous case, in the case where the proper mass is conserved.

In the case of a collision of a system of particles with another system, in which a certain amount of energy and momentum is transferred from the first system to the second one, classical arguments will lead to the same equivalent relation between the mass and energy

$$\Delta m = \Delta E/c^2$$

With the help of (4.6) we can construct an angular momentum in four-dimensional representation

$$M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\nu$$

which satisfies with the help of (4.19)

$$\Delta M_{\mu\nu}/\Delta\tau = (x_\mu + \Delta x_\mu)f_\nu - (x_\nu + \Delta x_\nu)f_\mu$$

which is clearly covariant.

## 5. *Classical Electrodynamics in Empty Space*

5.1. *Maxwell Equations in Discrete Form.* It is well known that Maxwell equations are consistent with the principles of special relativity. If we want to introduce the discrete representation of this equations, assumption (3) tells us that we have to change the differential expressions into difference equations. At this moment, one observation is very important. The finite increment of the independent variable should take an arbitrary integral value, not just the value unity, as is customarily said in the calculus of difference equations. We have followed this procedure in the previous sections, in all the formulas of discrete relativistic mechanics. The reason for it is that, under a Lorentz transformation, not only the variables but also their increments take different values in different inertial frames, and we cannot keep the unit value for the finite increments in any inertial frame. Therefore, the following substitutions should be made:

In one variable function, $f(x)$

$$\frac{df}{dx} \to \frac{\Delta f}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} \tag{5.1}$$

In more than one variable function, $f(x, y)$, for instance

$$\frac{\partial f}{\partial x} \to \frac{\Delta_x f}{\Delta x} = \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x} \tag{5.2}$$

Defining the discrete operators

$$\text{div}\,\mathbf{A} \equiv \frac{\Delta_x A_x}{\Delta x} + \frac{\Delta_y A_y}{\Delta y} + \frac{\Delta_z A_z}{\Delta z} \tag{5.3}$$

$$\text{rot}\,\mathbf{A} \equiv \left(\frac{\Delta_y A_z}{\Delta y} - \frac{\Delta_z A_y}{\Delta z}, \frac{\Delta_z A_x}{\Delta z} - \frac{\Delta_x A_z}{\Delta x}, \frac{\Delta_x A_y}{\Delta x} - \frac{\Delta_y A_x}{\Delta y}\right) \tag{5.4}$$

we can write Maxwell equations in discrete form

$$\operatorname{div} \mathbf{H} = 0, \qquad \operatorname{rot} \mathbf{E} + \frac{1}{c} \frac{\Delta_t \mathbf{H}}{\Delta t} = 0 \tag{5.5}$$

$$\operatorname{div} \mathbf{E} = \rho, \qquad \operatorname{rot} \mathbf{H} - \frac{1}{c} \frac{\Delta_t \mathbf{E}}{\Delta t} = \frac{\rho \mathbf{u}}{c} \tag{5.6}$$

where $\rho$ is the charge density and $\mathbf{u} = \Delta x/\Delta t$ the velocity with which the charges move. From (5.6) it is easy to derive the continuity equation in discrete form:

$$\frac{\Delta_t \rho}{\Delta t} + \operatorname{div} (\rho \mathbf{u}) = 0 \tag{5.7}$$

5.2. *The Electromagnetic Field Tensor and the Four-Current Density.* If we want to write (5.5) and (5.6) in covariant form, let us introduce the electromagnetic field tensor $F_{\mu\nu}$, and the four-current density $s_\mu$

$$F_{\mu\nu} = -F_{\nu\mu}, \qquad \epsilon_{kmn} F^{mn} = H_k, \qquad F_{4k} = E_k \tag{5.8}$$

$$s_k = (1/c)\rho u_k, \qquad s_4 = \rho \tag{5.9}$$

then, (5.5) and (5.6) can be written

$$\frac{\Delta_\mu F_{\nu\rho}}{\Delta x_\mu} + \frac{\Delta_\nu F_{\rho\mu}}{\Delta x_\nu} + \frac{\Delta_\rho F_{\mu\nu}}{\Delta x_\rho} = 0 \tag{5.10}$$

$$\frac{\Delta_\nu F_{\mu\nu}}{\Delta x_\nu} = s_\mu \tag{5.11}$$

where $\Delta_\mu$ is an abbreviation for $\Delta x_\mu$, but no summation over $\mu$ is understood. The continuity equation (5.7) reads

$$\frac{\Delta_\mu s_\mu}{\Delta x_\mu} = 0 \tag{5.12}$$

where summation over $\mu$ is understood.

5.3. *The Four-Potential.* The discrete form of the Maxwell equations is also consistent with the introduction of the four-potential. In fact, **H** and **E** can be written, as a consequence of (5.5), as follows:

$$\mathbf{H} = \operatorname{rot} \mathbf{A}, \qquad \mathbf{E} = -\operatorname{grad} \phi - \frac{1}{c} \frac{\Delta_t A}{\Delta t} \tag{5.13}$$

where **A** and $\phi$ can be chosen in such a way that they satisfy the Lorentz condition in discrete form

$$\operatorname{div} \mathbf{A} + \frac{1}{c} \frac{\Delta_t \phi}{\Delta t} = 0 \tag{5.14}$$

If we write the four-potential in covariant form

$$A_\mu = (\mathbf{A}, \phi) \tag{5.15}$$

(5.13) and (5.14) can be written

$$F_{\mu\nu} = \frac{\Delta_\mu A_\nu}{\Delta x_\mu} - \frac{\Delta_\nu A_\mu}{\Delta x_\nu} \tag{5.16}$$

$$\frac{\Delta_\mu A_\mu}{\Delta x_\mu} = 0 \tag{5.17}$$

and these expressions are invariant under a gauge transformation

$$A_\mu \to A'_\mu = A_\mu + \frac{\Delta_\mu \psi}{\Delta x_\mu} \tag{5.18}$$

Inserting (5.18) in (5.17), we have the condition

$$\frac{\Delta^2_{\mu\mu}\psi}{\Delta x_\mu \Delta x_\mu} = 0 \tag{5.19}$$

and finally, with the help of (5.16) and (5.17), (5.11) we can write Maxwell equation in the form

$$\frac{\Delta_{\mu\mu} A_\nu}{\Delta x_\mu \Delta x^\mu} = -s_\nu \tag{5.20}$$

5.4. *The Electromagnetic Energy-Momentum Tensor.* With the help of the electromagnetic field tensor it is possible to express covariantly the effect of a given field $F_{\mu\nu}$ on a moving particle of charge $e$. By classical arguments, consisten with the assumption of discreteness, one obtains

$$F_\mu = (e/c) F_{\mu\nu} U^\nu \tag{5.21}$$

For a distribution of charged matter with a four-density $s_\mu = \rho^0 U_\mu / c$, one gets a four-force density

$$f_\mu = F_{\mu\nu} s^\nu \tag{5.22}$$

With the help of the electromagnetic field tensor $F_{\mu\nu}$, we can construct, as in the continuous case, the Lorentz invariants

$$\tfrac{1}{2} F_{\mu\nu} F^{\mu\nu} = \mathbf{H}^2 - \mathbf{E}^2 \tag{5.23}$$

and

$$\epsilon_{\mu\nu\sigma\tau} F^{\mu\nu} F^{\sigma\tau} = \mathbf{H} \cdot \mathbf{E}$$

as well as the second-rank symmetric tensor, the electromagnetic energy-momentum tensor,

$$S_{\mu\nu} = F_{\mu\rho} F_\nu^{\ \rho} - \tfrac{1}{4} g_{\mu\nu} F_{\sigma\tau} F^{\sigma\tau} \tag{5.24}$$

Nevertheless, in the discrete case, the partial differences with respect to the four coordinates, contracted with the second index, do not give the four-force density. Instead we have

$$\frac{\Delta_\nu S_{\mu\nu}}{\Delta x_\nu} = F_{\mu\rho} s^\rho + \frac{\Delta_\nu F_{\mu\rho}}{\Delta x_\nu}(F_\nu{}^\rho + \Delta_\nu F_\nu{}^\rho) - \frac{1}{2} g_{\mu\nu} \frac{\Delta_\nu F_{\sigma\tau}}{\Delta x_\nu}(F^{\sigma\tau} + \Delta_\nu F^{\sigma\tau}) \quad (5.25)$$

5.5. *Monochromatic Plane Waves.* We now try to solve the Maxwell equations in discrete form in the case of electromagnetic field in empty space. If we choose the four potential $A_\mu$ such that $\phi = 0$, then we have

$$\mathbf{H} = \text{rot } \mathbf{A}, \qquad \mathbf{E} = -\frac{1}{c}\frac{\Delta_t \mathbf{A}}{\Delta t} \quad (5.26)$$

where rot **A** is a difference operator (5.4), the Lorentz condition (5.14) reads

$$\text{div } \mathbf{A} = 0 \quad (5.27)$$

and the wave equation (5.20) is given by

$$\frac{\Delta_{xx}\mathbf{A}}{(\Delta x)^2} + \frac{\Delta_{yy}\mathbf{A}}{(\Delta y)^2} + \frac{\Delta_{zz}\mathbf{A}}{(\Delta z)^2} - \frac{1}{c^2}\frac{\Delta_{tt}\mathbf{A}}{(\Delta t)^2} = 0 \quad (5.28)$$

We try a particular solution in the form of a plane wave (which is equivalent to use separation of variables)

$$\mathbf{A} = \mathbf{A}_0 e^{i(\mathbf{K}\cdot\mathbf{x} - \omega t)} \quad (5.29)$$

here $\mathbf{A}_0$ is a constant complex vector, $\omega$ is the angular frequency and **K** is the wave vector. Inserting (5.29) in (5.28) we see that **A** is a solution of the wave equation if the following condition is satisfied:

$$\left(\frac{e^{iK_x\Delta x} - 1}{\Delta x}\right)^2 + \left(\frac{e^{iK_y\Delta y} - 1}{\Delta y}\right)^2 + \left(\frac{e^{iK_z\Delta z} - 1}{\Delta z}\right)^2 - \left(\frac{e^{i\omega\Delta t} - 1}{c\Delta t}\right)^2 = 0 \quad (5.30)$$

Given some discrete increments $\Delta x$, $\Delta y$, $\Delta z$, $\Delta t$, the wave vector $(\mathbf{K}, \omega)$ will depend on them.

But we still have an extra condition, if we want (5.28) to be covariant under Lorentz transformations. Obviously (5.29) is a covariant expression because the phase $(\mathbf{K}\cdot\mathbf{x} - \omega t)$ is a Lorentz invariant. Therefore, we must require that (5.30) have the same form in an arbitrary inertial frame. The problem is solved if we make

$$K_x \Delta x = K_y \Delta y = K_z \Delta_z = \omega \Delta t = M \quad (5.31)$$

where $M$ is an arbitrary constant. In fact, suppose that in some inertial frame (5.31) is satisfied or, in covariant form

$$K_\mu = M/\Delta x_\mu \quad (5.32)$$

We go to another inertial frame. We have from (3.30) and (5.31)

$$K'_\mu = \Lambda_\mu{}^\nu K_\nu, \qquad \frac{1}{\Delta x'_\mu} = \Lambda_\mu{}^\nu \frac{1}{\Delta x_\nu} \tag{5.33}$$

Combining the last expression with (5.32) we get

$$\frac{M}{\Delta x'_\mu} = \Lambda_\mu{}^\nu \frac{M}{\Delta x_\nu} = \Lambda_\mu{}^\nu K_\nu = K'_\mu$$

or

$$K'_x \Delta x = K'_y \Delta x' = K'_z \Delta z' = \omega' \Delta t' = M$$

Therefore the numerators in (5.30) do not change; substituting (5.32) in the denominator we obtain

$$K_\mu K^\mu = K^2 - \omega^2 = 0 \tag{5.34}$$

which is an invariant expression.

The condition (5.32) is a very strong one, because it fixes the direction of the wave vector $K_\mu$ in the direction of the $[\Delta x_\mu]^{-1}$, which are given by the difference equation. This situation does not happen in the differential wave equations, because the increments $\Delta x_\mu$ always go to zero.

The covariance of (5.30) is also trivially satisfied if

$$M = 2\pi m \qquad (m = 0, 1, 2, \ldots) \tag{5.35}$$

but in this case we do not have (5.34).

If we substitute the solution (5.29) with the condition (5.32) in equations (5.26) and (5.27) we obtain, with suitable constant $M$,

$$\mathbf{E} = -(\omega/c)\mathbf{A}, \qquad \mathbf{H} = \mathbf{K} \times \mathbf{A} \tag{5.36}$$

$$\mathbf{K} \cdot \mathbf{A} = 0 \tag{5.37}$$

with the same interpretation as in the continuous case.

5.6. *Electromagnetic Waves in a Finite Volume.* In the last section nothing was said about the initial or boundary conditions of the wave equation and about the stability of the solution. If we require that the vector potential be bounded in some finite volume of space and that it satisfies the wave equation only in the grid points, $(r\Delta x, s\Delta y, t\Delta z)$ $r, s, t = 0, 1, 2, 3, \ldots$, then the use of the finite Fourier series will solve the problem.

The finite Fourier series (see for instance Milne, 1949; Forsythe and Wasov, 1960; Jordan, 1965) is based on the following properties of the exponential or trigonometric functions:

$$\sum_{x=1}^{N-1} \exp\left(i\frac{2\pi n}{N}x\right) = 0, \qquad n < N \tag{5.38}$$

where $x$, $n$, and $N$ are positive integers. Suppose a complete function $f(x)$ is defined in the lattice points $x = 0, 1, \ldots, N$. Then

$$f(x) = \sum_{n=0}^{N-1} a_n \exp\left(i\frac{2\pi n}{N}x\right) \tag{5.39}$$

The coefficients of this finite sum can be calculated multiplying both sides by $\exp\,[-i(2\pi m/N)_x]$ and doing the summation from $x = 0$ to $x = N - 1$,

$$\sum_{x=0}^{N-1} f(x) \exp\left(-i\frac{2\pi m}{N}x\right) = \sum_{n=0}^{N-1}\left(\sum_{x=0}^{N-1} a_n \exp\left(i\frac{2\pi(m-n)}{N}x\right)\right) \tag{5.40}$$

If $m \neq n$, the sum in brackets will give zero from (5.38). If $m = n$, the sum on the right side will give $Na_n$. Therefore

$$a_n = \frac{1}{N}\sum_{x=0}^{N-1} f(x) \exp\left(-i\frac{2\pi n}{N}x\right) \tag{5.41}$$

Assume now for simplicity that the boundary conditions of the solution of the wave equation (5.28) are

$$\begin{aligned}
&\mathbf{A}(0, y, z, t) = \mathbf{A}(a, y, z, t) = 0\\
&\mathbf{A}(x, 0, z, t) = \mathbf{A}(x, b, z, t) = 0\\
&\mathbf{A}(x, y, 0, t) = \mathbf{A}(x, y, c, t) = 0\\
&\mathbf{A}(x, y, z, 0) = \mathbf{A}(x, y, c, T) = \mathbf{A}_0(x, y, z)
\end{aligned} \tag{5.42}$$

with $a = N\Delta x$, $b = N\Delta y$, $c = N\Delta z$

The first three conditions give immediately in the general solution (5.29) the characteristic values of the wave vector:

$$K_x = \frac{2\pi n_x}{a}, \qquad K_y = \frac{2\pi n_y}{b}, \qquad K_z = \frac{2\pi n_z}{c} \tag{5.43}$$

where $n_x, n_y, n_z$ are arbitrary integer numbers. Therefore (5.29) can be written as an expansion of the partial waves

$$\mathbf{A}(x, y, z, t) = \Sigma\,\mathbf{B}(K_xK_yK_z)e^{i(\mathbf{K}\cdot\mathbf{x} - \omega t)} \tag{5.44}$$

where $\mathbf{K}$ are given by (5.43).

Now we impose the last condition of (5.42) on this expansion, in order to calculate the coefficient $\mathbf{B}(K_x, K_y, K_z)$. We must remember that the function $\mathbf{A}(x, y, z)$ is only defined in the grid points of the parallelepiped, namely, $(r\Delta x, r\Delta y, t\Delta z)$ $(r, s, t = 0, 1, 2, \ldots, N)$. Then the sum has only a finite number of terms. Another way to look at this cutoff in the series (5.44) is the following: If we want the wave equation to be covariant we concluded $K_\mu = M/\Delta x_\mu$.

Comparing with (5.43) we have

$$\frac{2\pi n_x}{N\Delta x} = \frac{M}{\Delta x}, \qquad \frac{2\pi n_y}{N\Delta y} = \frac{M}{\Delta y}, \qquad \frac{2\pi n_z}{N\Delta z} = \frac{M}{\Delta z} \tag{5.45}$$

But if $M = 2\pi m$, with $m$ an integer, the solution is trivial. This corresponds to the value of $n_x = n_y = n_z = N$, which are exactly the limit points of the series.

Finally, using the method described before (Milne, 1949; Forsythe and Wasov, 1960; Jordan, 1965), we calculate the coefficients in (5.44) by the finite sums:

$$\mathbf{B}(K_x, K_y, K_z) = \sum_{r=0}^{N-1} \sum_{s=0}^{N-1} \sum_{t=0}^{N-1} \mathbf{A}_0(r\Delta x, s\Delta y, t\Delta z) \times \exp\left[-i(K_x \Delta xr + K_y \Delta ys + K_z \Delta zt)\right] \tag{5.46}$$

## 6. *A Stronger Assumption: The Rational Character of The Physical Magnitudes*

In this paper, we have only used the hypothesis that all the space-time variables should take an integral value, and, as a consequence, all the increments of these variables can take also integral values.

It is possible to make a stronger assumption, as I have proposed recently (Lorente, 1974). In this paper, the physical magnitudes such as the proper time, four-momentum, the electromagnetic field, and the four-potential are allowed to take any real or complex value. We can go further and impose on these magnitudes the condition of taking only rational values. This assumption does not follow logically from the discreteness of the space-time variables, but it is consistent with it.

Now consider the expression for the proper time (3.18). It is easy to prove (Lorente, 1974, p. 235) that if we make

$$\begin{aligned} \Delta x &= -2mr(m^2 + q^2) + 2ms(mn - pq) - 2mt(mq + nq) \\ \Delta y &= -2mr(mn + pq) - 2ms(m^2 + p^2) + 2mt(mq - pn) \\ \Delta z &= 2mr(mr - nq) - 2ms(mq + np) - 2mt(m^2 + n^2) \\ \Delta t &= m^2(m^2 + n^2 + p^2 + q^2 + r^2 + s^2 + t^2) + (nt + ps + qr)^2 \end{aligned} \tag{6.1}$$

with $m, n, p, q, r, s, t$ arbitrary integral numbers, the proper time will be a rational number, given by

$$\Delta\tau = (1/c)[m^2(m^2 + n^2 + p^2 - r^2 - s^2 - t^2) - (nt + ps + qr)^2] \tag{6.2}$$

provided $c$ is given as a rational number. The expressions for the relativistic momentum and energy (4.3) will also be rational with the help of (6.2) and the condition that $m_0$ be rational.

All the arguments in Sections 3 and 4 can be carried out in similar way with this stronger assumption. In Section 5 we can also require that the fields take only rational values. This restriction, in the case of the solution of the wave equation, forces the vector potential and electromagnetic field to be proportional to one of the rational periodic functions. I have proved (Lorente, 1974, p. 237) that only four nontrivial circular functions are rational, namely, those real components of the quaternionic functions

$$\begin{aligned} \psi_2(x) &= (-1)^x \\ \psi_3(x) &= \left(\frac{-1 + ni + pj + qk}{2m}\right)^x, && 3m^2 = n^2 + p^2 + q^2 \\ \psi_4(x) &= \left(\frac{ni + pj + qk}{m}\right)^x, && m^2 = n^2 + p^2 + q^2 \\ \psi_6(x) &= \left(\frac{m + 3ni + 3pj + 3qk}{2m}\right)^x, && m^2 = 3(n^2 + p^2 + q^2) \end{aligned} \tag{6.3}$$

where $(i, j, k)$ is a quaternion basis, and $x, m, n, p, q$ can take only integral values. These functions must be used instead of the exponential functions (5.29). They also satisfy an equivalent condition (5.30) and (5.32).

As in the case of the proper time, now the rationality condition constrains the possible direction of the wave vector; in fact, since the frequency $\nu$ must be proportional to the norm of the wave vector $\mathbf{K}$, or what is equivalent, the period $T$ should be proportional to the wavelength $\lambda$, only those directions are allowed that are parallel to the vector $\mathbf{r}$ with components

$$\begin{aligned} r_1 &= m^2 - n^2 - p^2 + q^2 \\ r_2 &= 2mn + 2pq \\ r_3 &= -2mp + 2nq \\ r &= m^2 + n^2 + p^2 + q^2 \end{aligned} \tag{6.4}$$

with $m, n, p, q$ arbitrary integers (see Lorente, 1974, p. 244).

Therefore we have

$$T = jr, \qquad \lambda/c = jr \tag{6.5}$$

where $j$ is an arbitrary positive integer. To describe completely our rational periodic function we recall that the exponential form $\psi(t) = \exp(2\pi t/T)$ is periodic for $t = T$, $2T$, etc. In the same fashion we must write from (6.3)

$$\psi_l(t) = (\omega^2/|\omega|^2)^{lt/T} \qquad (l = 2, 3, 4, 6) \tag{6.6}$$

because for $t = T$, $2T$, etc. $\psi_l(t) = 1$. So our wave function will read

$$\psi_l(\omega^2/|\omega|^2)^{K\cdot x}$$

with

$$\mathbf{K} = \frac{l}{\lambda}\frac{\mathbf{r}}{r} = \frac{\mathbf{r}l}{jr^2}, \qquad K_4 = \frac{l}{T} = \frac{l}{jr}$$

The finite Fourier series is based now on the property of the functions (6.3)

$$\sum_{x=1}^{l=1} \left(\frac{\omega}{|\omega|^2}\right)(m/l)x = 0, \qquad m < l$$

and similar arguments can be carried out to those of Section 5.6.

## 7. *Concluding Remarks*

One may argue that the Lorentz group of the model is too small. Certainly, the Lorentz transformations under which the discreteness of the lattice is conserved become very reduced. Moreover, in the case of pure rotations only six matrices transform the whole lattice into itself (Lorente, 1974, p. 240). However, there is no contradiction between these drastic restrictions and observations. The model is postulated at a subquantum level: There is no indication yet about the order of magnitude of the size of the elementary cell. Therefore, all the classical and quantum predictions remain unchanged, because at this level space-time is continuous and the Lorentz group needs no restriction. From the theoretical point of view one can approximate continuous phenomena with a discrete structure of the world. For instance, in the case of scattering of two particles, given the observed angle between the incident and the outgoing particle, a particular transformation to the other rotated system can be approximated by rational cosines, as close as possible, with the use of rational rotations given in Lorente (1974, p. 229) and described by the complex vector

$$Z_k = z^{2k}|z|^{2(p-k)}, \qquad k = 0, 1, 2, \ldots, p$$

where $z = m + in$, and $m, n, p, k$ can take only integral values. The same arguments can be applied to the measured energy of a particle at rest and in other arbitrary inertial systems. One can approximate as much as possible both experimental observations by a discrete Lorentz transformation described by the hypercomplex number (see Lorente, 1974, p. 231)

$$U_k = u^{2k}|u|^{2(j-k)}, \qquad k = 0, 1, 2, \ldots, j$$

where $u = m + en$ ($e^2 = 1$) and $j, k, m, n$ take only integral values.

In my opinion the difficulty can be solved by epistemological considerations. It is well known that theoretical models must be linked to experimental results by rules of correspondence (see for instance Nagel, 1961):

> It is clear, however, that if a theory is to explain experimental laws, it is not sufficient that its terms be only implicitly defined. Unless something further is added to indicate how its implicitly defined terms are related to ideas occurring in experimental laws, a theory cannot be significantly affirmed or denied and in any way is scientifically useless (Nagel, 1961, p. 93).

Starting from our model one could find some rule of correspondence by which there is a contradiction between the model and observations. Nevertheless, the restriction of the Lorentz group in the model makes sense only in a subquantum level. Even in the case of an authentic rule of correspondence, the precision of the model should be tempered with the statistical character of the observations:

> The general point that emerges from these examples is that, though theoretical concepts may be articulated with a high degree of precision, rules of correspondence coordinate them with experimental ideas that are far less definite. The haziness that surrounds such correspondence rules is inevitable, since experimental ideas do not have the sharp contours that theoretical notions possess (Nagel, 1961, p. 100).

In my opinion, there is not yet any correspondence rule of the model. This model may be useless so far, but not in contradiction with experiments.

In a similar model, recently proposed by K. Wilson (1974b), the discrete properties of the field variables are linked to the confinement of quarks. This suggests some possible rule of correspondence for our model. Investigations along this line are being carried out.

## *Acknowledgments*

The author wishes to express his gratitude to Professor R. Jackiw for valuable comments, and to Professor P. Roman for a critical reading of the manuscript. He also is thankful to the Center of Theoretical Physics at M.I.T. where this work was done, and to the G.I.F.T. for financial support.

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